

# A class of nonlinear random walks related to the Ornstein-Uhlenbeck process

S.A. Muzychka, K.L. Vaninsky

July 6, 2011

## Abstract

Contrary to the theory of Markov processes, no general theory exists for the so called nonlinear Markov processes. We study an example of “nonlinear Markov process” related to classical probability theory, merely to random walks. This model provides interesting phenomena (absent in classical Markov chains): continuum of stationary measures, conserved quantities, convergence to stationary classical random walks etc.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
1.1	Simple random walks on $\mathbb{Z}$ with “discrete gaussian” stationary measure . . . . .	2
1.2	Nonlinear walks and main results. . . . .	3
<b>2</b>	<b>Proofs</b>	<b>7</b>
2.1	Existence and uniqueness . . . . .	7
2.2	Stationary points and the conserved integral . . . . .	21
2.3	Convergence. . . . .	23

## 1 Introduction

Contrary to the theory of Markov processes, no general theory exists for the so called nonlinear Markov processes. Though a general definition of a

nonlinear Markov process was introduced by H.P. McKean, [4] in his study of various models of kinetic theory. Subsequently various authors, see [5],[6], considered limits of stochastic many particles systems which lead to processes of this special type.

Here we give an example of “nonlinear Markov process” which is close to classical probability theory, merely to random walks. It appears as a mathematical model of a market with two type of agents or participants, traditionally called bulls and bears. This model provides interesting phenomena (absent in classical Markov chains): continuum of stationary measures, conserved quantities, convergence to stationary classical random walks etc. It is important that our system has some relation to the Ornstein-Uhlenbeck process. This underlies main intuition and makes the system solvable.

### 1.1 Simple random walks on $\mathbb{Z}$ with “discrete gaussian” stationary measure

Consider a continuous time Markov chain  $\eta_t$  (simple random walk) on  $\mathbb{Z}$ . The intensity of the jumps  $n \rightarrow n + 1$ ,  $n \rightarrow n - 1$  are correspondingly

$$\lambda_n = e^{-c(n-L)}; \quad \mu_n = e^{c(n-M)},$$

where  $c > 0$  and  $L$  and  $M$  are real numbers. The chain is ergodic and reversible. The detailed balance equations

$$\pi(n)\lambda_n = \pi(n+1)\mu_{n+1} \tag{1.1}$$

for the stationary measure  $\pi$  have the unique solution

$$\pi(n) = \frac{1}{\Xi} e^{-c(n-s)^2}, \quad s = \frac{L+M}{2}. \tag{1.2}$$

The normalization factor  $\Xi = \Xi(s, c)$  is given by

$$\Xi(s, c) = e^{-cs^2} \Theta\left(\frac{cs}{i\pi}, \frac{ci}{\pi}\right),$$

where

$$\Theta(v, \tau) = \sum e^{2\pi i v n + \pi i \tau n^2}$$

is the Jacobi theta function, see [1] p.188.

In addition to  $s$  we introduce another variable  $d = \frac{L-M}{2}$ . It is interesting that the invariant measure  $\pi$ , which should depend on both parameters  $L$  and  $M$ , depends on  $s$  only.

Let us note that the invariant measure does not change under the following transformation of the jump rates

$$\lambda_n \rightarrow \lambda_n \beta(n), \mu_n \rightarrow \mu_n \beta(n-1),$$

where  $\beta(n)$  is an arbitrary positive function. This follows from the detailed balance equations for the invariant measure. In particular,  $\beta(n)$  can be chosen so that the mean drift becomes asymptotically linear

$$m(n) = \lambda_n - \mu_n \sim -Cn, \quad C > 0$$

as for the classical Ornstein-Uhlenbeck process. Remind that the Ornstein-Uhlenbeck process is the unique stationary gaussian Markov process on  $\mathbb{R}$ .

## 1.2 Nonlinear walks and main results.

Consider the vector-function

$$X(t) = (L(t), M(t), p_n(t), n \in \mathbb{Z})$$

with  $(2 + \infty)$  real functions on the interval  $t \in [0, \infty)$  and denote

$$\lambda_n(t) = \beta(n)e^{-c(n-L(t))}, \mu_n(t) = \beta(n-1)e^{c(n-M(t))}, \quad (1.3)$$

where  $c > 0$  is some constant.

The vector-function  $X(t)$  is defined by the following infinite system of ordinary differential equations

$$\frac{dp_n}{dt} = \lambda_{n-1}p_{n-1} - (\lambda_n + \mu_n)p_n + \mu_{n+1}p_{n+1}, n \in \mathbb{Z} \quad (1.4)$$

$$\frac{dL}{dt} = - \sum_{n \in \mathbb{Z}} p_n \lambda_n + C_\lambda \quad (1.5)$$

$$\frac{dM}{dt} = \sum_{n \in \mathbb{Z}} p_n \mu_n - C_\mu \quad (1.6)$$

together with the initial conditions  $L(0), M(0), p_n(0)$ . We will assume that

$$p_n(0) \geq 0, \sum_{n \in \mathbb{Z}} p_n(0) = 1.$$

Otherwise speaking,  $p_n(0)$  define the probability measure  $p(0)$  on  $\mathbb{Z}$ .

A priori,  $C_\lambda$  and  $C_\mu$  are some positive constants. If however, there exists at least one fixed point  $(L, M, \pi)$  for these equations, then

$$C_\lambda = \sum_{n \in \mathbb{Z}} \pi_n \lambda_n, C_\mu = \sum_{n \in \mathbb{Z}} \pi_n \mu_n.$$

Then  $\pi_n$  satisfy equations (for fixed  $L, M$ )

$$\lambda_{n-1} \pi_{n-1} - (\lambda_n + \mu_n) \pi_n + \mu_{n+1} \pi_{n+1} = 0,$$

which look exactly as Kolmogorov equations for stationary probabilities of the countable Markov chain. It is known (see [3], p. 59, th. 7.1) that the only  $l_1$ -solution of these equations is positive (up to some multiplicative constant). Thus  $\pi_n$  satisfy also the detailed balance equations (1.1). It follows that  $C_\lambda, C_\mu$  are equal

$$C_\lambda = \sum_{n \in \mathbb{Z}} \pi_n \lambda_n = \sum_{n \in \mathbb{Z}} \pi_n \mu_n = C_\mu.$$

Then using the variables  $s, d$  introduced above we rewrite (1.4 – 1.6) in the following form:

$$\begin{cases} p'_n(t) = e^{cd} [\beta(n-1)e^{c(-n+1+s)} p_{n-1} - (\beta(n)e^{c(-n+s)} + \beta(n-1)e^{c(n-s)}) p_n + \\ \quad + \beta(n)e^{c(-n-1+s)} p_{n+1}], \quad n \in \mathbb{Z}; \\ s'(t) = -\frac{1}{2} e^{cd} (\sum_{n \in \mathbb{Z}} p_n \beta(n) e^{c(-n+s)} - \sum_{n \in \mathbb{Z}} p_n \beta(n-1) e^{c(n-s)}); \\ d'(t) = -\frac{1}{2} e^{cd} (\sum_{n \in \mathbb{Z}} p_n \beta(n) e^{c(-n+s)} + \sum_{n \in \mathbb{Z}} p_n \beta(n-1) e^{c(n-s)}) + C_\lambda, \end{cases} \quad (1.7)$$

where we have assumed the existence of the fixed point. First two equations show that the trajectory of a pair  $(p, s)$  does not depend on  $d$ . This observation will help us in the proof of convergence.

*Remark.* Now we want to explain some market model, which is the source of this paper. Assume that on the integer lattice  $\mathbb{Z}$  all points of the interval  $(-\infty, b]$  are occupied by “bulls” who want to buy and the points on the

interval  $[b + 1, \infty)$  who want to sell. The boundary  $b = b(t)$  changes with time as follows. There are two Poisson arrival streams of demands: to buy with the rate  $\lambda_b$  and to sell with the rate  $\mu_b$ . When the buy demand arrives the boundary immediately moves  $b \rightarrow b + 1$ , and conversely. The parameters  $L$  and  $M$  reflect the opinion of bulls and bears correspondingly, concerning the fair price.

Define the Banach space  $\mathbf{B}$  of vector-functions  $p = \{p_n\}_{n \in \mathbb{Z}}$  with the norm

$$\|p\|_\alpha = \sum_{n \in \mathbb{Z}} |p_n| \exp\left(\frac{n^2}{2} + \alpha|n|\right), \quad \alpha \in \mathbb{R}.$$

Throughout this paper we assume that  $\beta(n)$  satisfies the following condition

$$\sup_{n \in \mathbb{Z}} \beta(n) < \infty. \quad (1.8)$$

**Theorem 1.** *For any initial conditions such that  $p(0) \in \mathbf{B}$  is the probability measure, the solution of the system (1.7) exists on the interval  $[0, \infty)$  and is unique in the space  $\mathbf{B} \times C^2([0, \infty)) = \{(p, L, M)\}$ .*

*Moreover, for any  $t$  the quantities  $p_n(t)$  define the probability measure  $p(t)$ , that is  $p_n(t) \geq 0$ ,  $\sum_n p_n(t) = 1$ .*

**Theorem 2.** *If  $C_\lambda \neq C_\mu$  there are no fixed points. If  $C_\lambda = C_\mu > 0$  the set of fixed points is a one parameter family  $\{(L_s, M_s, \pi_s(n))\}$ , which depends on the parameter  $s \in \mathbb{R}$ . It is given explicitly by*

$$\begin{aligned} \pi_s(n) &= \frac{1}{\Xi} e^{-c(n-s)^2}, \quad \Xi = \sum_{n \in \mathbb{Z}} e^{-c(n-s)^2}; \\ L_s &= s + \ln \left[ C_\lambda \left( \frac{\sum_l e^{-c(l-s)^2}}{\sum_k \beta(k) e^{-c(k-s)^2} e^{c(-k+s)}} \right) \right]; \\ M_s &= s - \ln \left[ C_\lambda \left( \frac{\sum_l e^{-c(l-s)^2}}{\sum_k \beta(k) e^{-c(k-s)^2} e^{c(-k+s)}} \right) \right]. \end{aligned}$$

*Moreover,  $s = \frac{L_s + M_s}{2}$ .*

**Theorem 3.** *If  $C_\lambda = C_\mu > 0$  then there is a conserved quantity (invariant of motion)*

$$K = K(X) = L + M + \sum_{n \in \mathbb{Z}} n p_n.$$

Any hypersurface defined by the value of  $K(X)$  contains exactly one fixed point.

Speaking otherwise, the conserved quantity makes our phase space a fiber bundle over the real line, where each fiber contains exactly one fixed point.

For the next theorem we need, besides condition (1.8), the following condition: there is a positive constant  $C > 0$  such that for all  $n \in \mathbb{Z}$

$$\inf_{n \in \mathbb{Z}} \beta(n) > 0, \quad \frac{1}{e} \beta(n+1) - \beta(n) < -C, \quad \frac{1}{e} \beta(n-1) - \beta(n) < -C. \quad (1.9)$$

This (very technical) assumption we will need only for proving convergence. Note that unfortunately this conjecture does not cover the case of linear drift, but  $\beta(n) \equiv 1$  satisfies (1.9).

**Theorem 4.** Assume condition (1.9). Then for any initial point  $X(0)$  such that the initial probability measure  $p(0) \in \mathbf{B}$  the solution converges to the unique fixed point on the hypersurface defined by the value of  $K(X(0))$ .

**Theorem 5.** For any initial conditions  $X(0)$  such that the initial probability measure  $p(0) \in \mathbf{B}$  there exists a random process  $\xi(t) = \xi(t, X(0)) \in \mathbb{Z}, t \in [0, \infty)$ , with probability measure  $P = P_{X(0)}$  on the set  $X(t)$  of trajectories such that

$$P(\xi(t) = n) = p_n(t).$$

A such that the  $k$ -dimensional distributions of  $\xi(t)$ , for  $k > 1$ , are defined in Markovian way by

$$P_{X(0)}(\xi(t_1) = n_1, \dots, \xi(t_k) = n_k) = p_{n_1}(t_1) P_{X(0)}(n_2, t_2 | n_1, t_1) \dots P_{X(0)}(n_k, t_k | n_{k-1}, t_{k-1}). \quad (1.10)$$

Under condition (1.9), the  $k$ -dimensional distributions of  $\xi(t)$  tend as  $t \rightarrow \infty$  to the corresponding  $k$ -dimensional distributions of the stationary Markov process  $\eta_t$  defined above.

Let us note that while proving Theorem 1, we construct a family  $P_{X(0)}(n, s | m, t)$ ,  $t < s, m, n \in \mathbb{Z}$ , of stochastic matrices satisfying the semigroup property. Thus the latter theorem is just the definition of the process  $\xi(t)$ , Formula (1.10) looks like it defines a time inhomogeneous Markov process, but in fact it does not, since the transition kernels  $P_{X(0)}(\cdot, \cdot | \cdot, \cdot)$  depend on the initial conditions.

**Acknowledgments.** KV would like to thank Yuri Suhov, Henry McKean and Raghy Varadhan for stimulating discussions. Both authors would like to thank Vadim Malyshev for his interest in this work.

## 2 Proofs

Everywhere we will omit the parameter  $c$  assuming  $c = 1$ . To simplify notation we denote the pair of functions  $L$  and  $M$  by  $Z(t) = (L(t), M(t))$ .

### 2.1 Existence and uniqueness

Here we will prove Theorem 1. The scheme of the proof is the following. Assuming that the continuous functions  $L(t), M(t)$  are given, we prove that the solution of (1.4) exists and is unique in the appropriate Banach space, moreover it has some necessary properties in this space. Then we substitute this solution to the equations (1.5-1.6), thus obtaining two ODE with two unknown functions, and prove that the solution of these two equations exists.

**Two Banach spaces.** Consider the Banach space  $B_\alpha^+$ , which consists of infinite sequences  $(\nu_k, k \in \mathbb{Z})$  of real numbers with the norm:

$$\|\nu\|_\alpha^+ = \sum_{k \in \mathbb{Z}} e^{\frac{k^2}{2} + \alpha|k|} |\nu_k|,$$

and the Banach space  $B_\alpha^-$  with the norm:

$$\|f\|_\alpha^- = \sum_{k \in \mathbb{Z}} e^{-\frac{k^2}{2} - \alpha|k|} |f_k|.$$

Everywhere below  $\alpha$  is an arbitrary fixed real number. Let us explain the meaning of these Banach spaces.  $B_\alpha^+$  is the space of admissible measures of the process.  $B_\alpha^-$  is the space of admissible functions. The natural duality between  $B_\alpha^+$  and  $B_\alpha^-$  is

$$\langle \nu, f \rangle = \sum_{n \in \mathbb{Z}} \nu_n f_n, \quad \nu \in B_\alpha^+, f \in B_\alpha^-.$$

It is easy to see that

$$|\langle \nu, f \rangle| \leq \|\nu\|_\alpha^+ \|f\|_\alpha^-.$$

The space of bounded operators, acting on  $B_\alpha^+$  and  $B_\alpha^-$ , we denote by  $\mathcal{L}(B_\alpha^+)$  and  $\mathcal{L}(B_\alpha^-)$  correspondingly. The operators are acting on  $B_\alpha^+$  from the right, and on  $B_\alpha^-$  from the left.

Finally note that for any  $\alpha_1, \alpha_2 \in \mathbb{R}$ , such that  $\alpha_1 > \alpha_2$ , the following inclusions hold  $B_{\alpha_1}^+ \subset B_{\alpha_2}^+$  and  $B_{\alpha_2}^- \subset B_{\alpha_1}^-$ . We will use these properties below.

**Transition probabilities.** Assume now that  $L(t)$  and  $M(t)$  are some fixed continuous functions on  $\mathbb{R}_+$ . We will prove that the Markov process, defined by the Kolmogorov equations (1.4) for  $p_n(t)$  exists and is unique in  $B_\alpha^+$ . Denote by  $P(t, s)$ ,  $t \leq s$  the family of its transition probability matrixes,  $H(t)$  - the infinitesimal matrix. Let  $H_0(t)$  and  $V(t)$  be a diagonal and off diagonal parts of  $H(t)$  correspondingly.

First we present a useful formula for the transition probabilities valid for a denumerable inhomogeneous continuous time Markov chain. Denote  $\Delta_k(t, s) = \{(s_1, \dots, s_k) \in \mathbb{R}^k : t \leq s_k \leq \dots \leq s_1 \leq s\}$  the  $k$ -dimensional simplex.

**Lemma 1.** *Let  $X_t$  be a continuous time inhomogeneous Markov chain with denumerable state space and the family of transition probability matrices  $P(t, s)$ , defined for  $0 \leq t \leq s < \infty$ . Denote by  $H(t)$  an infinitesimal matrix of  $X_t$ . Let  $H_0(t)$  and  $V(t)$  be a diagonal and off diagonal parts of  $H(t)$ , then for any  $t < s$  the series*

$$P(t, s) = e^{\int_t^s H_0(s) ds} + \sum_{k=1}^{\infty} \int_{\Delta_k(t, s)} e^{\int_t^{s_k} H_0(s) ds} V(s_k) e^{\int_{s_k}^{s_{k-1}} H_0(s) ds} \dots V(s_1) e^{\int_{s_1}^s H_0(s) ds} ds_k \dots ds_1 \quad (2.1)$$

is absolutely norm convergent for some norm  $\|\cdot\|$ , if  $\sup_{u \in [t, s]} \|V(u)\| < \infty$ .

*Proof.* Formally the series is obtained by the iteration of the following formula

$$P(t, s) - e^{\int_t^s H_0(s) ds} = \int_t^s P(t, z) V(z) e^{\int_z^s H_0(s) ds} dz.$$

Since all diagonal terms of  $e^{\int H_0(s) ds}$  do not exceed 1, then using  $\sup_{u \in [t, s]} \|V(u)\| < \infty$ , and the formula for the volume of the simplex we get the result.  $\square$

We want to prove that the corresponding series converges in  $B_\alpha^+$  and the matrices  $P(t, s)$  are stochastic and satisfy the Chapman-Kolmogorov equations. In order to do this we have to check that  $V(t)$  are bounded operators in  $B_\alpha^+$ . First we will prove a technical lemma which will explain the condition (1.8).

Consider the following infinite three-diagonal matrix



$$V = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots \\ \mu_{n-1} & 0 & \lambda_{n-1} & & \\ & \mu_n & 0 & \lambda_n & \\ & & \mu_{n+1} & 0 & \lambda_{n+1} \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

We will consider  $V$  as the operator acting on infinite sequences from the right and from the left.

**Lemma 2.** *There exists a sequence  $\{c_n\}, n \in \mathbb{Z}$   $c_n > 0$  such that  $V$  is a bounded operator in the Banach space with the norm*

$$\|x\| = \sum_{n \in \mathbb{Z}} c_n |x_n|$$

*if and only if  $\sup_{n \in \mathbb{Z}} \lambda_n \mu_{n+1} < \infty$ .*

*Proof.* We will prove this lemma for the case when  $V$  is acting from the right. For the action from the left the proof is similar.

**Necessity.** Let  $V$  be bounded. Then if  $e_n = \delta_{0,n}$

$$\frac{\|e_n V\|}{\|e_n\|} = \frac{\|\mu_n e_{n-1} + \lambda_n e_{n+1}\|}{\|e_n\|} = \frac{c_{n-1}}{c_n} \mu_n + \frac{c_{n+1}}{c_n} \lambda_n \leq \|V\| = \text{const.}$$

Whence we have a double inequality

$$\frac{\mu_{n+1}}{\|V\|} \leq \frac{c_{n+1}}{c_n} \leq \frac{\|V\|}{\lambda_n}.$$

This gives the necessary conclusion.

**Sufficiency.** Assume that  $\sup_{n \in \mathbb{Z}} \lambda_n \mu_{n+1} < \infty$ . A straightforward calculation shows that for

$$c_n = \sqrt{\frac{\mu_1 \dots \mu_n}{\lambda_0 \dots \lambda_{n-1}}},$$

we get

$$\frac{\|e_n V\|}{\|e_n\|} = \sqrt{\lambda_{n-1} \mu_n} + \sqrt{\lambda_n \mu_{n+1}} < \infty.$$

□

Applying this lemma to the our case we see that the condition (1.8) is spelling natural. Indeed

$$\lambda_n(t)\mu_{n+1}(t) = \beta^2(n)e^{L(t)-M(t)},$$

therefore the condition of Lemma 2 is equivalent to (1.8).

**Lemma 3.** *Consider the operator valued function  $V(t)$ , defined above. Then this function takes values in the set of bounded operators in  $\mathcal{L}(B_\alpha^+)$ , or in  $\mathcal{L}(B_\alpha^-)$ . Moreover, it is continuous and*

$$\|V(t)\|_\alpha^\pm \leq \text{const}(e^{-M(t)} + e^{L(t)})$$

for any  $t$ .

*Proof.* Consider an arbitrary vector  $\nu \in B_\alpha^+$ . We have

$$\begin{aligned} \|\nu V(t)\|_\alpha^+ &= \sum_{k \in \mathbb{Z}} e^{\frac{k^2}{2} + \alpha|k|} |(\nu V(t))_k| = \\ &= \sum_{k \in \mathbb{Z}} e^{\frac{k^2}{2} + \alpha|k|} |e^{-k+1+L(t)} \beta(k-1) \nu_{k-1} + e^{k+1-M(t)} \beta(k) \nu_{k+1}| \leq \\ &\leq \text{const} \sum_{k \in \mathbb{Z}} e^{\frac{k^2}{2} + \alpha|k| - k + 1 + L(t)} |\nu_{k-1}| + \text{const} \sum_{k \in \mathbb{Z}} e^{\frac{k^2}{2} + \alpha|k| + k + 1 - M(t)} |\nu_{k+1}| = \\ &= \text{const} \sum_{k \in \mathbb{Z}} (e^{\frac{(k+1)^2}{2} + \alpha|k+1| - k - 1 + 1 + L(t)} + e^{\frac{(k-1)^2}{2} + \alpha|k-1| + k - 1 + 1 - M(t)}) |\nu_k| = \\ &= \text{const} \sum_{k \in \mathbb{Z}} (e^{\frac{k^2}{2} + \frac{1}{2} + \alpha|k+1| + L(t)} + e^{\frac{k^2}{2} + \frac{1}{2} + \alpha|k-1| - M(t)}) |\nu_k| \leq \text{const}(e^{L(t)} + e^{-M(t)}) \|\nu\|_\alpha^+. \end{aligned}$$

Similar calculation for any  $f \in B_\alpha^-$  implies the inequality for  $\|V(t)\|_\alpha^-$ .

It remains to check that  $V$  is continuous in  $t$ . We will do it for the space  $B_\alpha^+$  only. For  $B_\alpha^-$  it can be verified along the same lines. As in the estimates above for any arbitrary nonzero  $\nu \in B_\alpha^-$ , and arbitrary  $t_1, t_2$  we have:

$$\|\nu V(t_1) - \nu V(t_2)\|_\alpha^+ \leq \text{const}(|e^{L(t_1)} - e^{L(t_2)}| + |e^{-M(t_1)} - e^{-M(t_2)}|) \|\nu\|_\alpha^+.$$

Together with the fact that  $L(t)$  and  $M(t)$  are continuous, this implies our statement.  $\square$

**Lemma 4.** *Let  $t \leq s$ . Then the series (2.1) converges in both norms of  $\mathcal{L}(B_\alpha^+)$  and  $\mathcal{L}(B_\alpha^-)$  and therefore defines the bounded operator. Moreover,*

$$\|P(t, s)\|_\alpha^\pm \leq \exp(\text{const}(s-t) \sup_{t \in [0, s]} (e^{-M(t)} + e^{L(t)})).$$

*Proof.* Let us prove the lemma for  $B_\alpha^+$ . Arguments for  $B_\alpha^-$  are exactly the same. First, note that Lemma 3 implies that for any  $t \leq s$

$$\sup_{t \in [0, s]} \|V(t)\|_\alpha^+ \leq \text{const} \cdot \sup_{t \in [0, s]} (e^{-M(t)} + e^{L(t)}). \quad (2.2)$$

Since  $H_0$  consists of negative numbers, then for any  $t_1 \leq t_2$  we have  $\|e^{\int_{t_1}^{t_2} H_0(s) ds}\|_\alpha^+ \leq 1$ . Using that the volume of the simplex  $\Delta_k(t, s)$  is  $\frac{(s-t)^k}{k!}$ , and the estimates above, we obtain:

$$\|P(t, s)\|_\alpha^+ \leq 1 + \sum_{k=1}^{\infty} \frac{(\sup_{[0, s]} \|V(t)\|_\alpha^+)^k (s-t)^k}{k!} = e^{(s-t) \sup_{[0, s]} \|V(t)\|_\alpha^+}.$$

□

**Approximation by finite Markov chains.** Lemma 4 states that for any  $t \leq s$  the operator  $P(t, s)$  is defined in the spaces  $B_\alpha^+$ . It remains to prove that they define a Markov process. For the proof we need to introduce new notation.

Define truncated Markov processes  $X^m$  as the restriction of  $X$  on  $[-m, m]$ . More exactly  $X^m$  has the infinitesimal rates

$$k \rightarrow k+1: \quad \lambda_k^m(t) = \beta(k) e^{-k+L(t)}, \quad k \in [-m, m-1];$$

$$k \rightarrow k-1: \quad \mu_k^m(t) = \beta(k-1) e^{k-M(t)}, \quad k \in [-m+1, m].$$

Let  $H^m = H^m(t)$  be the infinitesimal matrix of  $X^m$ . Similar to what we have done before we write  $H^m$  in the form  $H^m = H_0^m + V^m$ , where  $H_0^m, V^m$  are its diagonal and off diagonal parts. For  $X^m$  obviously holds the formula analogous to (2.1)

$$\begin{aligned} P^m(t, s) &= e^{\int_t^s H_0^m(s) ds} + \\ &+ \sum_{k=1}^{\infty} \int_{\Delta_k(t, s)} e^{\int_t^{s_k} H_0^m(s) ds} V^m(s_k) \dots V^m(s_1) e^{\int_{s_1}^s H_0^m(s) ds} ds_k \dots ds_1. \end{aligned}$$

**Lemma 5. 1.** For any  $\pi \in B_\alpha^+$  and any  $t \leq s$ :  $\pi P^m(t, s) \rightarrow_{m \rightarrow \infty} \pi P(t, s)$  in the sense of the norm  $\|\cdot\|_\alpha^+$ ;

2. For any  $\pi \in B_\alpha^-$  and any  $t \leq s$ :  $P^m(t, s)\pi \rightarrow_{m \rightarrow \infty} P(t, s)\pi$  in the norm  $\|\cdot\|_\alpha^-$ .

*Proof.* We will give a proof only for  $B_\alpha^+$ . For  $B_\alpha^-$  the proof is the same. Fix some  $0 \leq t \leq s$  and define

$$\begin{aligned} \Gamma^m(s_1, \dots, s_k) &:= \pi e^{\int_t^{s_k} H_0(s) ds} V(s_k) e^{\int_{s_k}^{s_{k-1}} H_0(s) ds} \dots V(s_1) e^{\int_{s_1}^s H_0(s) ds} - \\ &\quad - \pi e^{\int_t^{s_k} H_0^m(s) ds} V^m(s_k) e^{\int_{s_k}^{s_{k-1}} H_0^m(s) ds} \dots V^m(s_1) e^{\int_{s_1}^s H_0^m(s) ds}, \end{aligned}$$

where  $\{s_j\}_{j=1}^k \in \Delta_k(t, s)$ . It is easy to check that  $\pi e^{\int_{t_1}^{t_2} H_0^m(s) ds} \rightarrow_{m \rightarrow \infty} \pi e^{\int_{t_1}^{t_2} H_0(s) ds}$  in the sense of the norm  $\|\cdot\|_\alpha^+$  for  $t_1 \leq t_2$ , and  $\pi V^m(t) \rightarrow_{m \rightarrow \infty} \pi V(t)$ . Therefore in the norm

$$\Gamma^m(s_1, \dots, s_k) \rightarrow_{m \rightarrow \infty} 0$$

for all sets  $\{s_j\}_{j=1}^k$ , which belong to correspondent simplex. Let us estimate the difference of the  $k$ -th terms  $A_k$  and  $A_k^m$  of the series for  $\pi P(t, s)$  and  $\pi P^m(t, s)$  correspondingly

$$\|A_k - A_k^m\|_\alpha^+ \leq \int_{\Delta_k(t, s)} \|\Gamma^m(s_1, \dots, s_k)\|_\alpha^+ ds_k \dots ds_1.$$

Using the estimates, similar to the one used in Lemma 4, it is easy to check that  $\|\Gamma^m(s_1, \dots, s_k)\|_\alpha^+$  bounded on simplex, namely:

$$\|\Gamma^m(s_1, \dots, s_k)\|_\alpha^+ \leq 2\|\pi\|_\alpha^+ (\text{const} \cdot \sup_{[0, s]} (e^{-M(t)} + e^{L(t)}))^k.$$

Therefore, Lebesgue theorem implies that:

$$\int_{\Delta_k(t, s)} \|\Gamma^m(s_1, \dots, s_k)\|_\alpha^+ ds_k \dots ds_1 \rightarrow_{m \rightarrow \infty} 0,$$

i.e.

$$\|A_k - A_k^m\|_\alpha^+ \rightarrow_{m \rightarrow \infty} 0.$$

Moreover, using a formula for the volume of simplex it is easy to get, that:

$$\|A_k - A_k^m\|_\alpha^+ \leq \frac{s^k}{k!} 2\|\pi\|_\alpha^+ (\text{const} \cdot \sup_{[0, s]} (e^{-M(t)} + e^{L(t)}))^k,$$

i.e.  $\sum_{k=1}^\infty \|A_k - A_k^m\|_\alpha^+$  converges uniformly in  $m$ .

We will use the following simple

**Proposition 6.** *Let the series  $\sum_{k=1}^{\infty} a_{km}$  converge uniformly in  $m = 0, 1, \dots$ , and  $a_{km} \rightarrow_{m \rightarrow \infty} 0$ , then  $\sum_{k=1}^{\infty} a_{km} \rightarrow_{m \rightarrow \infty} 0$ .*

Using this proposition we have:

$$\|\pi P^m(t, s) - \pi P(t, s)\|_{\alpha}^+ \leq \sum_{k=1}^{\infty} \|A_k - A_k^m\|_{\alpha}^+ \rightarrow_{m \rightarrow \infty} 0.$$

Therefore  $\|\pi P^m(t, s) - \pi P(t, s)\|_{\alpha}^+ \rightarrow_{m \rightarrow \infty} 0$ .  $\square$

**Corollary 7.** *The matrices  $P(\cdot, \cdot)$  satisfy the Chapman-Kolmogorov equations.*

*Proof.* It is apparent that for all  $t \leq u \leq s$ , and  $m \in \mathbb{N}$ , the Chapman-Kolmogorov equations hold

$$P^m(t, u)P^m(u, s) = P^m(t, s).$$

Fix some  $\pi \in B_+^{\alpha}$ , then

$$\begin{aligned} \pi[P(t, u)P(u, s) - P(t, s)] &= \pi[(P(t, u) - P^m(t, u))P(u, s) + \\ &+ P^m(t, u)(P(u, s) - P^m(u, s)) + (P^m(t, s) - P(t, s))] = \\ \pi[(P(t, u) - P^m(t, u))P(u, s) + (P^m(t, u) - P(t, u))(P(u, s) - P^m(u, s)) + \\ &+ P(t, u)(P(u, s) - P^m(u, s)) + (P^m(t, s) - P(t, s))]. \end{aligned}$$

Using Lemma 5 and the uniform boundness in the norm  $P^m(t, s)$  on the segment  $[t, s]$  (easy to check), we obtain, in the limit  $m \rightarrow \infty$ , the required statement.  $\square$

**Corollary 8.** *The matrices  $P(\cdot, \cdot)$  are stochastic.*

*Proof.* Let  $h \in B_{\alpha}^{-}$  be the vector which consists of all 1's (i.e. for any  $i \in \mathbb{Z}$ ,  $h_i = 1$ ), then for any  $m \in \mathbb{N}$ ,  $t \leq s$  we have:

$$(P^m(t, s)h)_i = \sum_{j \in \mathbb{Z}} (P^m(t, s))_{ij} = 1 \implies P^m(t, s)h = h.$$

Using Lemma 5, we obtain in the norm

$$P^m(t, s)h \rightarrow P(t, s)h.$$

Since  $P^m(t, s)h = h$ , the latter formula implies:

$$P(t, s)h = h,$$

but this means that:

$$\sum_{j \in \mathbb{Z}} (P(t, s))_{ij} = 1.$$

□

*Remark.* In the Corollary 8 we used the fact that the chain is not exploding. In those cases when the trajectory runs to infinity it is impossible to adjust the norm  $\|\cdot\|_\alpha^-$  such that the vector  $h$ , which consists of all 1's belongs to this space. However, and in these cases the matrix  $P(\cdot, \cdot)$  can be defined, but it will not be stochastic.

**Corollary 9.** *The family  $P(t, s)$ ,  $t \leq s$  is continuous in  $t$  and  $s$  in  $\mathcal{L}(B_\alpha^\pm)$ .*

*Proof.* Since  $P(t, s)$  satisfies Kolmogorov-Chapman equations, it is enough to prove that  $P(t, t + t_1)$  is continuous at zero as a function of  $t_1$ . Using the formula (2.1) and the estimates analogous to those of Lemma 4, we have:

$$\begin{aligned} \|P(t, t + t_1) - \text{Id}\|_\alpha^\pm &\leq \|e^{\int_t^{t+t_1} H_0(s) ds} - \text{Id}\|_\alpha^\pm + \\ &+ (\exp(\text{const} \cdot t_1 \sup_{u \in [t, t+t_1]} (e^{-M(u)} + e^{L(u)})) - 1) \rightarrow_{t_1 \rightarrow 0} 0. \end{aligned}$$

This implies our statement. □

**Lemma 10.** *Distribution  $p(t) \in B_\alpha^+$  as a function of time is real analytic on  $\mathbb{R}_+$ . The solution of (1.4) is unique in the class of real analytic functions on  $\mathbb{R}_+$ .*

*Moreover, the solution of (1.4) is unique in the class of continuous functions  $p(t)$  in  $B_\alpha^+$ .*

*Proof.* It is easy to check that for the remainder  $R_n(t)$  of the series (??) we have  $\|R_n(t)\|_\alpha^+ = O(t^n)$ . Moreover, it is easy to check that  $p_k(t)$  is infinitely differentiable. These two statements imply the result.

Let  $p^1(t)$  and  $p^2(t)$  are two different analytic solutions of (1.4), then their difference  $p(t) = p^1(t) - p^2(t)$  is a solution of (1.4) with trivial initial data

$$p(0) \equiv 0.$$

Then it is easy to check using induction in  $l$ , that for any  $k \in \mathbb{Z}, l \in \mathbb{Z}_+$

$$p_k(0)^{(l)} = 0.$$

Then due to the condition of the lemma  $p_k(t) = 0$  for any  $k \in \mathbb{Z}$ .

The proof of the last assertion is similar to the calculations made in Lemma 1. Actually since

$$\frac{d}{dz} \left( p(z) e^{\int_z^t H_0(s) ds} \right) = p(z) V(z) e^{\int_z^t H_0(s) ds},$$

integrating from 0 to  $t$  we get

$$p(t) - p(0) e^{\int_0^t H_0(s) ds} = \int_0^t p(z) V(z) e^{\int_z^t H_0(s) ds} dz.$$

Now iterating this formula we conclude:

$$p(t) = p(0) e^{\int_0^t H_0(s) ds} + \sum_{k=1}^n \int_{\Delta_k(0,t)} p(0) e^{\int_0^{s_k} H_0(s) ds} V(s_k) \dots V(s_1) e^{\int_{s_1}^t H_0(s) ds} ds_k \dots ds_1 + R_{n+1},$$

where

$$R_{n+1} = \int_{\Delta_{n+1}(0,t)} p(s_{n+1}) V(s_{n+1}) \dots V(s_1) e^{\int_{s_1}^t H_0(s) ds} ds_{n+1} \dots ds_1$$

converges to 0. □

Thus, we constructed family  $P(\cdot, \cdot)$ , which consists of stochastic matrices and satisfies (1.4).

**Local existence and uniqueness of the nonlinear system.** In this section we consider the original problem with  $Z = (L, M)$  which satisfies the system of differential equations (1.4)-(1.6). From the formal point of view, to prove that the process is defined on the segment  $[0, T]$  ( $T \geq 0$  is arbitrary), it is necessary to solve the infinite system of differential equations for the pair  $(p, Z)$ . However, as it was shown above for any  $Z \in C[0, T] \times C[0, T]$ , there exists the unique Markov process  $X_Z$ , having transition probabilities  $P_Z(\cdot, \cdot)$ ,

the infinitesimal matrix  $H_Z(t)$  and the distribution  $p_Z(t)$  at the moment  $t$ . By substitution one can get a closed system of differential equations for  $Z$ :

$$\begin{cases} L'(t) = -p(0)P_Z(0, t)\lambda_Z(t) + C_\lambda, \\ M'(t) = +p(0)P_Z(0, t)\mu_Z(t) - C_\mu \end{cases} \quad (2.3)$$

with the initial data

$$\begin{cases} L(0) = L_0, \\ M(0) = M_0, \end{cases}$$

where  $\lambda_Z(t)$  and  $\mu_Z(t)$  are transition rates for  $X_Z(t)$ . Thus one takes out  $p(t)$  from consideration.

Introduce the necessary notation. Fix some  $R > \max(|L_0|, |M_0|)$ . Let  $B(T, R) = \{f \in C[0, T] : \max_{t \in [0, T]} |f(t)| \leq R\}$  be the closed ball in the space  $C[0, T]$  of continuous functions on  $[0, T]$ , equipped with the uniform metrics  $\rho_{B(T, R)}$ . We consider the space  $B(T, R)^2 = B(T, R) \times B(T, R)$  with the metrics

$$\rho(Z_1, Z_2) = \rho((L_1, M_1), (L_2, M_2)) = \rho_{B(T, R)}(L_1, L_2) + \rho_{B(T, R)}(M_1, M_2).$$

It will be convenient to take the parameters  $L_0, M_0$  equal to zero. This can be done by shifting the coordinates.

In the estimates below we will use some unknown functions of initial data  $L_0, M_0, R, T$  etc. By  $c(\dots)$  we denote any nonnegative function, nondecreasing in each of its arguments.

**Lemma 11.** *Let  $p(0) \in B_\alpha^+$ ,  $Z_1, Z_2 \in B(T, R)^2$ , then for any  $t \in [0, T]$*

$$\|p(0)P_{Z_1}(0, t) - p(0)P_{Z_2}(0, t)\|_{\alpha-1}^+ \leq c(R, T, \|p(0)\|_\alpha^+, |L_0|, |M_0|)\rho(Z_1, Z_2).$$

(The left side of the inequality is defined, since  $B_\alpha^+ \subset B_{\alpha-1}^+$ .)

*Proof.* First we prove that  $H_{Z_1}(t) - H_{Z_2}(t)$  is a family of bounded, continuous in  $t$  operators, acting from  $B_\alpha^+$  into  $B_{\alpha-1}^+$ . Let us estimate their norm. For arbitrary  $\nu \in B_\alpha^+$  we obtain:

$$\begin{aligned} \|\nu H_{Z_1}(t) - \nu H_{Z_2}(t)\|_{\alpha-1}^+ &\leq \|\nu V_{Z_1}(t) - \nu V_{Z_2}(t)\|_{\alpha-1}^+ + \\ &+ \|\nu H_{0Z_1}(t) - \nu H_{0Z_2}(t)\|_{\alpha-1}^+ := I_1(t) + I_2(t). \end{aligned}$$

Let us estimate each term separately.



**a.** We have

$$\begin{aligned}
I_1(t) &= \|\nu V_{Z_1}(t) - \nu V_{Z_2}(t)\|_{\alpha-1}^+ \leq \\
&\leq \sum_{k \in \mathbb{Z}} |\beta(k-1)e^{-k+1+L_1(t)+L_0}\nu_{k-1} + \beta(k)e^{k+1-M_1(t)-M_0}\nu_{k+1} - \\
&\quad - \beta(k-1)e^{-k+1+L_2(t)+L_0}\nu_{k-1} - \beta(k)e^{k+1-M_2(t)-M_0}\nu_{k+1}| e^{\frac{k^2}{2}+(\alpha-1)|k|} \leq \\
&\leq \text{const} \sum_{k \in \mathbb{Z}} |e^{L_1(t)+L_0} - e^{L_2(t)+L_0}| \cdot |e^{-k+1}\nu_{k-1}e^{\frac{k^2}{2}+(\alpha-1)|k|}| + \\
&\quad + \text{const} \sum_{k \in \mathbb{Z}} |e^{-M_1(t)-M_0} - e^{-M_2(t)-M_0}| \cdot |e^{k+1}\nu_{k+1}e^{\frac{k^2}{2}+(\alpha-1)|k|}| \leq \\
&\leq \text{const} \|\nu\|_{\alpha-1}^+ |e^{L_1(t)+L_0} - e^{L_2(t)+L_0}| + \text{const} \|\nu\|_{\alpha-1}^+ |e^{-M_1(t)-M_0} - e^{-M_2(t)-M_0}| \leq \\
&\leq c(R, |L_0|, |M_0|) \rho(Z_1, Z_2) \|\nu\|_{\alpha-1}^+ \leq c(R, |L_0|, |M_0|) \rho(Z_1, Z_2) \|\nu\|_{\alpha}^+.
\end{aligned}$$

**b.** Note, that  $H_{0Z_{1,2}}(t)$  are diagonal matrices, therefore

$$\begin{aligned}
I_2(t) &\leq \sum_{k \in \mathbb{Z}} |\beta(k)e^{-k+L_1+L_0} + \beta(k-1)e^{k-M_1-M_0} - \beta(k)e^{-k+L_2+L_0} - \beta(k-1)e^{k-M_2-M_0}| \cdot \\
&\quad \cdot |\nu_k| \cdot e^{\frac{k^2}{2}+(\alpha-1)|k|} \leq \text{const} \sum_{k \in \mathbb{Z}} |e^{L_1+L_0} - e^{L_2+L_0}| \cdot |\nu_k| \cdot e^{-k+\frac{k^2}{2}+(\alpha-1)|k|} + \\
&\quad + \text{const} \sum_{k \in \mathbb{Z}} |e^{-M_1-M_0} - e^{-M_2-M_0}| \cdot |\nu_k| \cdot e^{k+\frac{k^2}{2}+(\alpha-1)|k|} \leq \\
&\leq c(R, |L_0|, |M_0|) \rho(Z_1, Z_2) \|\nu\|_{\alpha}^+.
\end{aligned}$$

Using **a.** and **b.**, we have:

$$\|H_{Z_1}(t) - H_{Z_2}(t)\| \leq c(R, |L_0|, |M_0|) \rho(Z_1, Z_2), \quad (2.4)$$

where  $\|\cdot\|$  is a standart supremum norm. Continuity in  $t$  of  $H_{Z_1}(t) - H_{Z_2}(t)$  can be checked in the same way.

Now we are going to prove a useful inequality which we will employ later. Consider  $\Gamma(t) = P_{Z_1}(0, t)P_{Z_2}(t, T)$ ,  $t \in [0, T]$  and differentiate

$$\frac{d}{dt}\Gamma(t) = P_{Z_1}(0, t)(H_{Z_1}(t) - H_{Z_2}(t))P_{Z_2}(t, T).$$

Due to the results of section 1 the following sequence of transformations holds

$$B_{\alpha}^+ \longrightarrow_{P_{Z_1}(0, t)} B_{\alpha}^+ \longrightarrow_{H_{Z_1}(t) - H_{Z_2}(t)} B_{\alpha-1}^+ \longrightarrow_{P_{Z_2}(t, T)} B_{\alpha-1}^+.$$

Therefore  $\frac{d}{dt}\Gamma(t)$  is a family of bounded and continuous in  $t$  operators acting from  $B_\alpha^+$  into  $B_{\alpha-1}^+$ . Integrating from 0 to  $T$ , we obtain:

$$P_{Z_1}(0, T) - P_{Z_2}(t, T) = \int_0^T P_{Z_1}(0, t)(H_{Z_1}(t) - H_{Z_2}(t))P_{Z_2}(t, T)dt.$$

Then

$$\begin{aligned} I &:= \|p(0)P_{Z_1}(0, t) - p(0)P_{Z_2}(0, t)\|_{\alpha-1}^+ \leq \\ &\leq \|p(0)\|_\alpha^+ \int_0^T \|P_{Z_1}(0, t)\|_\alpha^+ \cdot \|H_{Z_1}(t) - H_{Z_2}(t)\| \cdot \|P_{Z_2}(t, T)\|_{\alpha-1}^+ dt. \end{aligned}$$

Using (2.4) and the estimate of Lemma 4, we get

$$I \leq c(R, T, \|p(0)\|_\alpha^+, |L_0|, |M_0|)\rho(Z_1, Z_2).$$

□

**Lemma 12.** *For any initial data  $L_0, M_0 \in \mathbb{R}$ ,  $p(0) \in B_\alpha^+$  the system (2.3) has a unique solution for  $t$  sufficiently small.*

*Proof.* First we verify that for any  $Z \in B(T, R)^2$  the function

$$\langle p(0)P_Z(0, t), \lambda_Z(t) \rangle : \mathbb{R} \rightarrow \mathbb{R}$$

is a function continuous on the segment  $[0, T]$ . From Lemma 9 it follows that  $p(0)P_Z(0, \cdot) : \mathbb{R} \rightarrow B_\alpha^+$  is a continuous function. Moreover for any  $t \in [0, T]$  and  $k \in \mathbb{Z}$

$$|(p(0)P_Z(0, t))_k| \leq \|p(0)\|_\alpha^+ \|P_Z(0, t)\|_\alpha^+ e^{-\frac{k^2}{2} - \alpha|k|},$$

and

$$|(p(0)P_Z(0, t))_k \lambda_{kZ}(t)| \leq \beta(k) e^{L(t)+L_0} \|p(0)\|_\alpha^+ \|P_Z(0, t)\|_\alpha^+ e^{-\frac{k^2}{2} - (\alpha-1)|k|}.$$

Therefore the series

$$\langle p(0)P_Z(0, t), \lambda_Z(t) \rangle = \sum_{k \in \mathbb{Z}} (p(0)P_Z(0, t))_k \lambda_{kZ}(t)_k$$

is majorized by a uniformly convergent series on  $[0, T]$ . That implies our statement. It is proved analogously that for any  $Z \in B(T, R)^2$   $\langle p(0)P_Z(0, \cdot), \mu_Z(\cdot) \rangle$  is a function continuous on  $[0, T]$ .

Now we can rewrite (2.3) in the integral form:

$$\begin{cases} L(t) = - \int_0^t [\langle p(0)P_Z(0, s), \lambda_Z(s) \rangle - C_\lambda] ds, \\ M(t) = \int_0^t [\langle p(0)P_Z(0, s), \mu_Z(s) \rangle - C_\mu] ds. \end{cases}$$

It is sufficient to check that for sufficiently small  $T$  the following map is contracting in  $B(T, R)^2$  :

$$F : \begin{pmatrix} L(t) \\ M(t) \end{pmatrix} \rightarrow \begin{pmatrix} - \int_0^t [\langle p(0)P_Z(0, s), \lambda_Z(s) \rangle - C_\lambda] ds \\ \int_0^t [\langle p(0)P_Z(0, s), \mu_Z(s) \rangle - C_\mu] ds \end{pmatrix}.$$

We have to check first that we can find such  $T$  that the above map maps  $B(T, R)^2$  into itself (we assume that  $R$  is fixed ).

Denote  $F_L, F_M$  the projections of  $F$  onto the first and second coordinates and estimate

$$\begin{aligned} |F_L(Z, t)| &\leq \int_0^t |\langle p(0)P_Z(0, s), \lambda_Z(s) \rangle| ds + tC_\lambda \leq \int_0^T \|p(0)P_Z(0, s)\|_\alpha^+ \|\lambda_Z(s)\|_\alpha^- ds + \\ &+ tC_\lambda \leq c(\|p(0)\|_\alpha^+, |L_0|, R) \int_0^T \|P_Z(0, s)\|_\alpha^+ ds + tC_\lambda \leq \\ &\leq Tc(\|p(0)\|_\alpha^+, |L_0|, R, T) + tC_\lambda. \end{aligned}$$

From this we see that we can find  $T$  such that for any  $t \in [0, T]$   $|F_L(Z, t)| \leq R$ . Analogous estimate can be obtained for  $F_M(Z, t)$ , therefore for sufficiently small  $T$  we obtain, that  $F_L(Z, t)$  and  $F_M(Z, t)$  can not leave  $[-R, R]$ , but this means exactly that  $F$  maps  $B(T, R)^2$  into itself.

Consider the difference

$$F_L(Z_1, t) - F_L(Z_2, t) = \int_0^t [\langle p(0)P_{Z_2}(0, s), \lambda_{Z_2}(s) \rangle - \langle p(0)P_{Z_1}(0, s), \lambda_{Z_1}(s) \rangle] ds.$$

Note that  $|F_L(Z_1, t) - F_L(Z_2, t)| \leq \int_0^t (I(s) + J(s)) ds$ , where

$$I(t) = |\langle p(0)P_{Z_2}(0, t), \lambda_{Z_2}(t) - \lambda_{Z_1}(t) \rangle|,$$

$$J(t) = |\langle p(0)P_{Z_2}(0, t) - p(0)P_{Z_1}(0, t), \lambda_{Z_1}(t) \rangle|.$$

First we estimate  $I(t)$ . Note that

$$\lambda_{Z_2}(t) - \lambda_{Z_1}(t) = (e^{L_2(t)} - e^{L_1(t)})\xi,$$

where  $\xi \in B_\alpha^-$  is the vector with the components  $\xi_n = \beta(n)e^{-n+L_0}$ . Then

$$I(t) \leq e^{L_2(t)-L_1(t)} \|p(0)P_{Z_2}(0, t)\|_\alpha^+ \|\xi\|_\alpha^- \leq c(T, R, |L_0|, |M_0|, \|p(0)\|_\alpha^+) \rho(Z_1, Z_2).$$

Now let us estimate  $J(t)$ . From Lemma 11 we conclude that

$$\begin{aligned} J(t) &\leq \|p(0)P_{Z_2}(0, t) - p(0)P_{Z_1}(0, t)\|_{\alpha-1}^+ \|\lambda_{Z_1}(t)\|_{\alpha-1}^- \leq \\ &\leq c(|L_0|, R) \|p(0)P_{Z_2}(0, t) - p(0)P_{Z_1}(0, t)\|_{\alpha-1}^+ \leq \\ &\leq c(R, T, \|p(0)\|_\alpha^+, |L_0|, |M_0|) \rho(Z_1, Z_2). \end{aligned}$$

Thus, we have

$$|F_L(Z_1, t) - F_L(Z_2, t)| \leq c(R, T, \|p(0)\|_\alpha^+, |L_0|, |M_0|) t \rho(Z_1, Z_2).$$

Analogous expression we get for  $|F_M(Z_1, t) - F_M(Z_2, t)|$ . Then for sufficiently small  $T$  (remind that  $c(\dots)$  is nondecreasing function in each argument) we obtain that the map is contracting.  $\square$

Using Lemma 12 and the contraction, we obtain existence and uniqueness of the solution of the system (2.3) for small  $t$ . It remains to prove that this solution can be extended to the entire axes.

**Global existence.** In the previous section we have proved the local existence and uniqueness of our process. In this section we will prove that the process can be extended to all  $\mathbb{R}_+$ . For this purpose it is sufficient to prove that  $L(t)$  and  $M(t)$  can not run off to the infinity. In other words there is no explosion in our model.

**Lemma 13.** *There exist positive nondecreasing functions  $f_1$  and  $f_2$  defined on  $\mathbb{R}_+$ , such that*

$$\begin{aligned} -f_1(t) &\leq L(t) \leq L_0 + C_\lambda \cdot t; \\ M_0 - C_\mu \cdot t &\leq M(t) \leq f_2(t). \end{aligned}$$

*Proof.* Note that integrating (2.3) we get the first assertion:

$$L(t) \leq L_0 + C_\lambda \cdot t; \quad M(t) \geq M_0 - C_\mu \cdot t;$$

Therefore using (2.2)

$$\|P_Z(0, t)\|_\alpha^+ \leq \text{const} \cdot \exp(te^{C_\mu \cdot t - M_0} + te^{C_\lambda \cdot t - L_0}) := f(t).$$

We see that  $f(t)$  is a positive nondecreasing function on  $\mathbb{R}_+$ . Applying (2.3) we have

$$L'(t) \geq -\|p(0)P_Z(0, t)\|_\alpha^+ \|\xi\|_\alpha^- e^{L(t)} + C_\lambda \geq -Cf(t)e^{L(t)} + C_\lambda,$$

where  $C > 0$  is an arbitrary sufficiently large constant, and  $\xi = \{e^{-n}\}$ . This inequality implies that if  $L(\cdot) \leq \ln \frac{C_\lambda}{Cf(\cdot)}$ , thus

$$L'(\cdot) \geq 0.$$

Taking  $C$  large enough and such that  $L(0) \geq \ln \frac{C_\lambda}{Cf(0)}$  we conclude

$$L(t) \geq \ln \frac{C_\lambda}{Cf(t)} := f_1(t).$$

Analogous estimates can be made for  $M(t)$ . □

The Theorem 1 is proved.

## 2.2 Stationary points and the conserved integral

Here we will prove Theorem 2. We already saw that if at least one fixed point exists then  $C_\lambda = C_\mu$ . Now we will find the fixed points explicitly.

Let  $(\pi, L, M)$  is a fixed point of  $X$ . Note that  $L, M$  are real numbers. The invariant measure  $\pi$  can be uniquely identified with our discrete gaussian measure, introduced above

$$\pi_n = \frac{1}{\Xi} e^{-(n-s)^2},$$

where  $\Xi = \sum_{n \in \mathbb{Z}} e^{-(n-s)^2}$ .

Thus we are left with the following two equations

$$\begin{cases} \sum_{k \in \mathbb{Z}} \frac{1}{\Xi} e^{-(k-s)^2} \beta(k) e^{-k+L} - C_\lambda = 0; \\ \sum_{k \in \mathbb{Z}} \frac{1}{\Xi} e^{-(k-s)^2} \beta(k-1) e^{k-M} - C_\mu = 0. \end{cases}$$

Let us rewrite the first equation in terms of  $s$  and  $d$

$$C_\lambda = \sum_{k \in \mathbb{Z}} \frac{1}{\Xi} e^{-(k-s)^2} \beta(k) e^{-k+L} = \sum_{k \in \mathbb{Z}} \frac{e^{-(k-s)^2}}{\Xi} \beta(k) e^{-k+s+d} = e^d \frac{\sum_k \beta(k) e^{-(k-s)^2} e^{-k+s}}{\sum_l e^{-(l-s)^2}}.$$

The expression for  $d$  follows

$$d = \ln \left[ C_\lambda \left( \frac{\sum_l e^{-(l-s)^2}}{\sum_k \beta(k) e^{-(k-s)^2} e^{-k+s}} \right) \right].$$

Then

$$L = s + d = s + \ln \left[ C_\lambda \left( \frac{\sum_l e^{-(l-s)^2}}{\sum_k \beta(k) e^{-(k-s)^2} e^{-k+s}} \right) \right]$$

and similarly

$$M = s - \ln \left[ C_\lambda \left( \frac{\sum_l e^{-(l-s)^2}}{\sum_k \beta(k) e^{-(k-s)^2} e^{-k+s}} \right) \right].$$

Now we will prove Theorem 3. Namely, we will show that the system of equations (1.4,1.5,1.6) has the following integral of motion

$$K(t) = 2s(t) + \sum_{k \in \mathbb{Z}} k p_k(t) = L(t) + M(t) + \sum_{k \in \mathbb{Z}} k p_k(t). \quad (2.5)$$

Summing up equations for  $L$  and  $M$ , we have

$$(L + M)'(t) = - \sum_{k \in \mathbb{Z}} p_k(t) \lambda_k(t) + \sum_{k \in \mathbb{Z}} p_k(t) \mu_k(t) = - \left( \sum_{k \in \mathbb{Z}} k p_k(t) \right)'.$$

It remains to prove that  $(\sum_{k \in \mathbb{Z}} k p_k(t))' = \sum_{k \in \mathbb{Z}} p_k(t) (\lambda_k(t) - \mu_k(t))$ . Using estimates (2.8) (we will prove it below), it is easy to see that the series  $\sum_{k \in \mathbb{Z}} k p_k(t)$  can be differentiated term by term. Thus:

$$\begin{aligned} \left( \sum_{k \in \mathbb{Z}} k p_k(t) \right)' &= \sum_{k \in \mathbb{Z}} k (p_{k-1} \lambda_{k-1} - (\lambda_k p_k + \mu_k p_k) + \mu_{k+1} p_{k+1}) = \\ &= \sum_{k \in \mathbb{Z}} \{ ((k+1) - k) \lambda_k p_k + (-k + (k-1)) \mu_k p_k \} = \sum_{k \in \mathbb{Z}} p_k(t) (\lambda_k - \mu_k). \end{aligned}$$

## 2.3 Convergence.

Here we will prove Theorem 4 assuming both conditions on  $\beta(n)$ , introduced above, namely (1.8) and (1.9).

We will define two Lyapunov functions. The first of them  $Q(t)$  will be positive and decreasing along the trajectory outside some special set. The second  $W(t)$  decreases along a trajectory everywhere, but can take big negative values. Using these two functions we can prove the convergence.

**Boundness of  $L$  and  $M$ .** Note that the pair of equations for  $L$  and  $M$  are equivalent to the following pair of equations

$$\begin{cases} s'(t) = -\frac{1}{2}e^d \left( \sum_{n \in \mathbb{Z}} p_n \beta(n) e^{-n+s} - \sum_{n \in \mathbb{Z}} p_n \beta(n-1) e^{n-s} \right); \\ d'(t) = -\frac{1}{2}e^d \left( \sum_{n \in \mathbb{Z}} p_n \beta(n) e^{-n+s} + \sum_{n \in \mathbb{Z}} p_n \beta(n-1) e^{n-s} \right) + C_\lambda. \end{cases} \quad (2.6)$$

We will use the following

**Proposition 14.** *If the terms of the absolutely convergent series  $u(x) = \sum_{n=1}^{\infty} u_n(x)$  are continuously differentiable on the segment  $[a, b]$  and the series of derivatives  $\sum_{n=1}^{\infty} u'_n(x)$  converges uniformly in  $(a, b)$ , then*

$$\frac{d}{dx} \sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} u'_n(x).$$

In order to use this statement we have to make additional estimates. If  $p(0) \in B_\alpha^+$ , then  $\|p(t)\|_\alpha^+ \leq \text{const}$  for any sufficiently small segment  $[0, T]$ . This implies for any  $t \in [0, T]$

$$|p_k(t)| \leq e^{-\frac{k^2}{2} - \alpha|k|} \|p_k(t)\|_\alpha^+ \leq \text{const} \cdot e^{-\frac{k^2}{2} - \alpha|k|}. \quad (2.7)$$

From this, using the formula for  $p'_k(t)$ , it is easy to see that for any  $t \in [0, T]$

$$|p'_k(t)| \leq \text{const} e^{-\frac{k^2}{2} - (\alpha-1)|k|}. \quad (2.8)$$

Moreover, we will need the following Grownall type result

**Proposition 15.** *Let  $f(\cdot)$  be some differentiable function on  $[0, \infty)$  such that*

$$f'(\cdot) \geq (\leq) g(\cdot)(C_1 - C_2 f(\cdot)), \quad C_2 > 0$$

*where  $g(\cdot)$  is a positive function. Then  $f(\cdot)$  is bounded from below (above).*

Define the first Lyapunov function (compare with  $d'(t)$ )

$$Q(t) = \sum_{n \in \mathbb{Z}} p_n(t) (\beta(n)e^{-n+s} + \beta(n-1)e^{n-s}).$$

**Lemma 16.** *1.  $Q(t)$  is bounded. 2.  $|\sum_{k \in \mathbb{Z}} kp_k(t) - s(t)|$  is bounded.*

*Proof.* **1.** Using estimate (2.8), it is easy to check, that the series for  $Q'(t)$  can be differentiated term by term. First we find  $Q'(t)$

$$Q'(t) = \sum_{n \in \mathbb{Z}} p'_n (\beta(n)e^{-n+s} + \beta(n-1)e^{n-s}) + \sum_{n \in \mathbb{Z}} p_n (\beta(n)e^{-n+s} - \beta(n-1)e^{n-s}) s'.$$

Using (??) we note that the second component of the expression is less than 0. So

$$Q'(t) \leq \sum_{n \in \mathbb{Z}} p'_n (\beta(n)e^{-n+s} + \beta(n-1)e^{n-s}) = e^{-d} \sum_{n \in \mathbb{Z}} p'_n(t) (\lambda_n(t) + \mu_n(t)).$$

Using Kolmogorov's equations and opening the brackets we have

$$Q'(t) \leq e^{-d} \sum_{n \in \mathbb{Z}} p_n(t) \{ \lambda_n \lambda_{n+1} + \mu_n \mu_{n-1} + \lambda_n \mu_{n+1} + \mu_n \lambda_{n-1} - 2\lambda_n \mu_n - \lambda_n^2 - \mu_n^2 \}.$$

Substituting expressions (1.3) for  $\lambda_n$  and  $\mu_n$  we have:

$$\begin{aligned} Q'(t) &\leq e^d \sum_{n \in \mathbb{Z}} p_n(t) [\beta(n)\beta(n+1)e^{-2n-1+2s} + \beta(n-1)\beta(n-2)e^{2n-1-2s} + e\beta^2(n) + \\ &+ e\beta^2(n-1) - 2\beta(n-1)\beta(n) - \beta^2(n)e^{-2n+2s} - \beta^2(n-1)e^{2n-2s}] := e^d \sum_{n \in \mathbb{Z}} S_n(s)\pi_n(t). \end{aligned}$$

We state that for the just defined function  $S_n(s)$

$$S_n(s) + \beta(n)e^{-n+s} + \beta(n-1)e^{n-s} \leq \text{const.}$$

Indeed, putting  $x := e^{s-n}$ , we can rewrite this inequality in the following form:

$$\beta(n) \left[ \left( \frac{1}{e} \beta(n+1) - \beta(n) \right) x^2 + x \right] + \beta(n-1) \left[ \left( \frac{1}{e} \beta(n-2) - \beta(n-1) \right) x^2 + x \right] \leq \text{const.}$$



Therefore conditions (1.8) and (1.9) imply required inequality. Now we get

$$Q'(t) \leq e^d \sum_{n \in \mathbb{Z}} p_n(t) (\text{const} - \beta(n)e^{-n+s} - \beta(n-1)e^{n-s}) = e^d (\text{const} - Q(t)).$$

Using Proposition 15, we obtain that  $Q$  is bounded.

**2.** We have

$$\begin{aligned} \left| \sum_{n \in \mathbb{Z}} n p_n(t) - s(t) \right| &= \left| \sum_{n \in \mathbb{Z}} n p_n - s \right| = \sum_{n \in \mathbb{Z}} |p_n(n-s)| \leq \\ &\leq \text{const} \sum_{n \in \mathbb{Z}} |p_n| (\beta(n)e^{-n+s} + \beta(n-1)e^{n-s}) = \text{const} \cdot Q(t). \end{aligned}$$

It remains to use the boundness of  $Q$ . □

**Lemma 17.** *Functions  $L$  and  $M$  are bounded.*

*Proof.* Note that this statement is equivalent to the fact that  $s$  and  $d$  are bounded. Boundness of  $s$  directly follows from formula (2.5) and part 2 of Lemma 16. Boundness of  $d$  is proved as Proposition 15, since

$$d'(t) = -\frac{1}{2}e^d Q(t) + C_\lambda.$$

□

**Relative entropy for constant  $L$  and  $M$ .** In the nonlinear case we will prove convergence using the relative entropy method. In this subsection we will introduce auxiliary notions and lemmas for constant  $Z$ .

If  $Z = \text{const}$  there exists only one invariant measure  $\pi$ , given by (1.2). Define the entropy of the distribution  $p = p(t)$  relative to  $\pi$  in the following way

$$H(t) = H(p(t)|\pi) = \sum_{n \in \mathbb{Z}} p_n \ln \frac{p_n}{\pi_n} = \sum_{n \in \mathbb{Z}} \pi_n \varphi \left( \frac{p_n}{\pi_n} \right), \quad (2.9)$$

where  $\varphi(x) = x \ln x$ .

*Remark.* As the factor  $\Xi$  adds a constant to  $H(t)$ , everywhere below we will assume  $\Xi = 1$ . Thus  $p(t)$  is just a finite (not necessary probability) measure.

The fact that  $H(t)$  decreases in time is known [2]. We will show that the series (2.9) is convergent and can be differantated term by term. We will use the following technical lemma:

**Lemma 18.** *For any  $\varepsilon > 0$  there exists  $C=C(\varepsilon) > 0$  such that for  $t > \varepsilon$*

$$\ln p_n(t) \geq -Ce^{|n|}(1+t).$$

*Proof.* Let  $n \in \mathbb{Z}_+$ ,  $p_0(0) > 0$ . We will make a very rough estimate. It is evident that  $p_n(t)$  is greater than the product of the following probabilities:

- The probability  $P_1$  that at the moment  $t = 0$  the particle is at the point 0.
- The probability  $P_2$  that the only jumps before time  $t$  are as follows: from 0 to 1, from 1 to 2, etc., from  $n-1$  to  $n$ .

Otherwise speaking

$$\begin{aligned} p_n(t) &\geq p_0(0) \cdot \left[ \prod_{k=0}^{n-1} \left( \frac{\beta(n)e^{-n+L}}{\beta(n-1)e^{n-M} + \beta(n)e^{-n+L}} \right) \right] \cdot \left[ e^{-(\beta(n-1)e^{n-M} + \beta(n)e^{-n+L})t} \right] \\ &\cdot \left[ \frac{(\min_n (\beta(n-1)e^{n-M} + \beta(n)e^{-n+L}) t)^n}{n!} e^{-\min_n (\beta(n-1)e^{n-M} + \beta(n)e^{-n+L})t} \right] \geq \\ &\geq \pi_0(0) C^n e^{-n(n-1)} \cdot e^{-Ce^n t} \cdot \frac{(ct)^n}{n!} e^{-ct}. \end{aligned}$$

Taking the logarithm, we get the requested assertion. Similar calculation can be made in the case  $p_0(0) = 0$ ,  $p_k(0) \neq 0$ ,  $k \neq 0$ , and  $n \in \mathbb{Z}^-$ .  $\square$

**Corollary 19.** *The series (2.9) is well defined for  $t \geq 0$ , and for  $t > 0$  it can be differentiated term by term.*

*Proof.* Whereas (2.7) it is easy to see that the series (2.9) converges on any sufficiently small interval. Using Lemma 18 we get that the corresponding series of derivatives uniformly converges on any interval enough small. The corollary implies Proposition 14.  $\square$

**Lemma 20.** *For  $t > 0$*

$$\frac{d}{dt} H(t) \leq 0.$$

*The equality is attained as soon as  $p(t) = \pi$  up to multiplicative factor.*

**Nonlinear case.** Now we will consider the case when  $Z = (L, M)$  satisfy equations (1.5,1.6). Similarly we define

$$\pi_n^s = e^{-(s-n)^2}$$

and the relative entropy

$$H(t) = H(p(t)|\pi^s) = \sum_{n \in \mathbb{Z}} p_n \ln \frac{p_n}{\pi_n^s} = \sum_{n \in \mathbb{Z}} p_n (\ln p_n + (s-n)^2). \quad (2.10)$$

The measure  $\pi^s$  is not stochastic because we prefer not to normalize it.

First of all we have to check that the series (2.10) converges and can be differentiated term by term. Taking into account that  $L$  and  $M$  are bounded we can prove the technical lemma similar to Lemma 18 :

**Lemma 21.** *Let  $\varepsilon > 0$  be a fixed number. Then there exists  $C = C(\varepsilon) > 0$  such that for  $t > \varepsilon$*

$$\ln p_n(t) \geq -C e^{|n|} (1+t).$$

**Corollary 22.** *The series (2.10) is well defined for  $t \geq 0$ , and for  $t > 0$  it can be differentiated term by term.*

Define

$$W(t) = H(t) + 2Ks(t) - 3s^2(t),$$

where  $K = 2s + \sum_{n \in \mathbb{Z}} n p_n(t) = \text{const}$  is the invariant, introduced above.

**Lemma 23.** *For  $t > 0$*

$$\frac{d}{dt} W(t) \leq 0.$$

*The equality is attained as soon as  $p(t) = \pi^s$  up to multiplicative factor.*

*Proof.* For  $t > 0$  the series (2.10) can be differentiated term by term. Thus

$$\frac{d}{dt} H(t) = \frac{\partial H}{\partial \pi} \cdot \frac{d\pi}{dt} + \frac{\partial H}{\partial s} \cdot \frac{ds}{dt}. \quad (2.11)$$

The first term of the right side represents a derivative of the relative entropy for fixed  $s$ , therefore, using results of the preceding subsection, we conclude that this term is negative. Let us calculate the second term

$$\frac{\partial H}{\partial s} \cdot \frac{ds}{dt} = \sum_{n \in \mathbb{Z}} 2p_n(s-n)s' = 2ss' \sum_{n \in \mathbb{Z}} p_n - 2s' \sum_{n \in \mathbb{Z}} p_n n = 2ss' - 2s' \sum_{n \in \mathbb{Z}} np_n(t).$$

Using the invariant  $K$  we get

$$\frac{\partial H}{\partial s} \cdot \frac{ds}{dt} = 2ss' - 2s'(K - 2s) = 6ss' - 2s'K = (3s^2 - 2Ks)'$$

and by (2.11) we get

$$\frac{d}{dt}\{H + 2Ks - 3s^2\} \leq 0.$$

□

**Corollary 24.** *Let  $p(0) \in B_\alpha^+$  and  $L_0, M_0, \alpha \in \mathbb{R}$  be arbitrary numbers. Then the convergence holds.*

*Proof.* In Lemma 23 we introduced the function  $W = W(p, Z)$ , which can be considered as the Lyapunov function. Therefore the proof has a quite standart scheme.

First,  $p(t)$  belongs to a bounded (supremum norm) closed subset of the set  $C_0(\bar{\mathbb{Z}})$  of functions  $f$  on  $\bar{\mathbb{Z}} = \mathbb{Z} \cup \{\infty\}$ , continuous at infinity and such that  $f(\infty) = 0$ . That is

$$\{p(t)\}_{t \in \mathbb{R}_+} \subset B_\alpha^+ \cap \{\mu = \{\mu_n\} : \mu_n \geq 0, \sum_{n \in \mathbb{Z}} \mu_n = 1\} \subset C_0(\bar{\mathbb{Z}}).$$

It follows that the domain of  $W$  lies in a compact subset of  $C_0(\bar{\mathbb{Z}}) \times \mathbb{R}^2$ , as  $Z = (L, M)$  is bounded.

Therefore the trajectory  $\{(p(t), Z(t))\}$  has at least one limiting point. Let  $(\pi^*, Z^*)$  be one of such points. As  $W$  decreases along the trajectory

$$\frac{d}{dt}W_{(\pi^*, Z^*)}(t) = 0.$$

This follows from continuous differentiability of  $W(t)$ . Using Lemma 23 we conclude that  $\pi_n^* = e^{-(s^*-n)^2}$  up to some factor.

Using the invariant it is easy to check that  $s^*$  is defined uniquely. In the introduction we mentioned that the trajectory of  $(p, s)$  does not depend on the choice of  $d$ . Therefore  $(p, s)$  converges to  $(\pi^*, s^*)$ . It remains to show

that  $d$  converges to  $d^*$ . Remind that we can rewrite the equation for  $d$  in the following form

$$d' = -\frac{1}{2}e^d Q(t) + C_\lambda,$$

where

$$Q(t) = \sum_{n \in \mathbb{Z}} p_n \beta(n) e^{-n+s} + \sum_{n \in \mathbb{Z}} p_n \beta(n-1) e^{n-s}.$$

Note that due to the established convergence of  $s$  and  $p$

$$Q(t) \rightarrow_{t \rightarrow \infty} Q^* = \text{const.}$$

We have an ordinary differential equation which can be solved explicitly

$$e^{-d(t)} = C_1 e^{-C_\lambda \cdot t} + \frac{1}{2} e^{-C_\lambda \cdot t} \int_0^t e^{C_\lambda \cdot s} Q(s) ds.$$

As  $Q(t)$  converges we obtain the required result.  $\square$

## References

- [1] A. Hurwitz and R. Courant. Theory of functions. 1968. Moscow.
- [2] Th. Liggett. Interacting particle systems. 2005. Springer.
- [3] Kai Lai Chung. Markov chains with stationary transition probabilities. 1960. Springer.
- [4] H. McKean. A class of Markov processes associated with nonlinear parabolic equations. Proceedings of National Ac. Sci. USA., 1966, v.56, No. 6, 1907-1811.
- [5] A.-S, Sznitman. Topics in propagation of chaos. Lecture Notes Math., 1991, v. 1464, 165-251.
- [6] C. Graham, S. Meleard. Stochastic particle approximations for generalized Boltzmann models and convergence estimates. Ann. Prob., 1997, v. 25, No. 1, 115-132.
- [7] V.A. Malyshev and A.D. Manita Dynamics of Phase Boundary with Particle Annihilation. MPRF, 2009, pp. 575-584.

- [8] A. Veretennikov. On ergodic measures for McKean–Vlasov stochastic equations, In: Monte Carlo and Quasi-Monte Carlo Methods 2004 Niederreiter, H; Talay, D. (Eds.), (2006), 471-486.
- [9] A. Rybko, S. Shlosman. Poisson Hypothesis for Information Networks (A study in non-linear Markov processes) I. Domain of Validity. arXiv:math/0406110.
- [10] Alexander Rybko, Senya Shlosman. Poisson Hypothesis for information networks (A study in non-linear Markov processes). arXiv:math-ph/0303010.
- [11] N. Vvedenskaya, Y. Suhov, V. Belitsky. A non-linear model of limit order book dynamics. arXiv:1102.1104v1 [math.PR] 5 Feb 2011.
- [12] M. Nagasawa, H. Tanaka. On the Propagation of Chaos for Diffusion Processes with Drift Coefficients Not of Average Form. Tokyo J. Math. Vol. 10, No. 2, 1987.
- [13] Р. Л. Добрушин. Уравнения Власова. Функциональный анализ и его приложения, т. 13, вып. 2, 1979, 48—58.

K. Vaninsky  
 Department of Mathematics  
 Michigan State University  
 East Lansing, MI 48824  
 USA vaninsky@math.msu.edu

S. Muzychka  
 Faculty of Mathematics and Mechanics  
 Moscow State University  
 Vorobjevy Gory  
 Moscow, Russia